Correspondence

Determination of Equivalent Circuit Parameters*

At a given frequency, a lossless discontinuity on a transmission line may be represented by an equivalent circuit consisting of an ideal transformer and two lengths of transmission line.[1] With reference to Fig. 1, let

n = turns ratio of ideal transformer.
θ_1, θ_2 = electrical lengths of transmission lines.
ϕ_1 = electrical distance of electric field null from reference plane T_1.
ϕ_2 = electrical distance of short circuit from reference plane T_2.
Z_1 = normalized characteristic impedance of output transmission line.

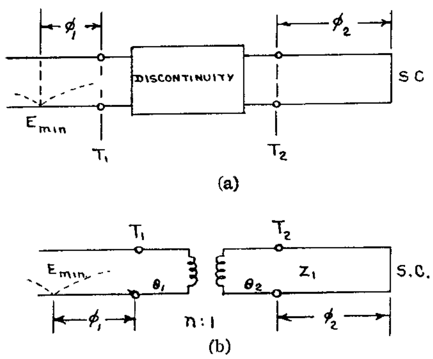

Fig. 1—A lossless discontinuity and equivalent circuit.

The field minimum position ϕ_1 is related to the short-circuit position ϕ_2 by a bilinear transformation of the form

$$\tan\phi_1 = \frac{A + B\tan\phi_2}{C + D\tan\phi_2}. \quad (1)$$

An analysis of the equivalent circuit of Fig. 1(b), shows that the field minimum position is given by

$$\tan(\phi_1 + \theta_1) = -N^2 \tan(\phi_2 + \theta_2) \quad (2)$$

where $N^2 = n^2 Z_1$.

This expression can be cast into the form of (1).

The analysis to follow will derive relations between θ_1, θ_2, N, and A, B, C, D. A plot of ϕ_1 vs ϕ_2 from (2) yields a curve of the form illustrated in Fig. 2. The curve is symmetrical about a line of slope -1 and periodic with a period π. If $N<1$, the slope of the curve at P_1 is $-N^2$ and at P_2 is $-(1/N^2)$. Also at P_2,

$$\phi_1 + \theta_1 = +n\pi + \frac{\pi}{2}$$

and

$$\phi_2 + \theta_2 = s\pi + \frac{\pi}{2}$$

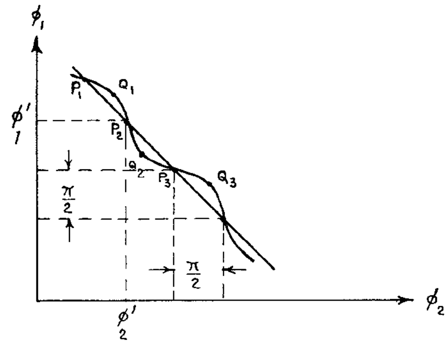

Fig. 2—Typical curve of ϕ_1 vs ϕ_2.

where n, s are suitable integers. If $N>1$, the slope at P_2 is $-N^2$ and $\phi_1+\theta_1=n\pi$, $\phi_2+\theta_2=s\pi$ at P_2. These results will be used later in the analysis.

If (1) is differentiated and the derivative $d\phi_1/d\phi_2$ equated to -1, one gets

$$1 + \tan^2\phi_1 = \frac{(AD - BC)(1 + \tan^2\phi_2)}{(C + D\tan\phi_2)^2}. \quad (3)$$

Eliminating ϕ_1 by means of (1) gives

$$(D^2 + B^2 - AD + BC)\tan^2\phi_2 + (2CD + 2AB)\tan\phi_2 + (C^2 + A^2 + BC - AD) = 0. \quad (4)$$

This equation may be solved for ϕ_2 and determines the points Q_1, Q_2, Q_3 in Fig. 2. The solution is of the form

$$\tan\phi_2 = \alpha \pm \beta. \quad (5)$$

Let ϕ_{21} and ϕ_{22} be the two values of ϕ_2 obtained from (5). If these two values of ϕ_2 are averaged, either the point P_2 or P_3 is determined. One has

$$\tan(\phi_{21} + \phi_{22}) = \frac{\tan\phi_{21} + \tan\phi_{22}}{1 - \tan\phi_{21}\tan\phi_{22}} = \frac{2\alpha}{1 - \alpha^2 + \beta^2}. \quad (6)$$

When α and β are determined from (4), one gets

$$\phi_2' = \frac{\phi_{21} + \phi_{22}}{2} = \frac{1}{2}\tan^{-1}\frac{2(AB + CD)}{A^2 + C^2 - B^2 - D^2}. \quad (7)$$

As yet it is not known whether ϕ_2' corresponds to the point P_2 or P_3. However, if one takes arbitrarily, $\theta_2 = -\phi_2' + s\pi$, then from (1)

$$\phi_1' = \tan^{-1}\frac{A + B\tan\phi_2'}{C + D\tan\phi_2'} \quad (8)$$

and

$$\theta_1 = -\phi_1' + n\pi.$$

From the results presented earlier, it is noted that at the point where $\phi_1+\theta_1=n\pi$ and $\phi_2+\theta_2=s\pi$, the slope of the curve is always $-N^2$. Thus, equating the derivative $d\phi_1/d\phi_2$ to $-N^2$ at the point ϕ_1', ϕ_2' gives

$$n^2 Z_1 = N^2 = \frac{(AD - BC)(1 + \tan^2\phi_2')}{(C + D\tan\phi_2')^2(1 + \tan^2\phi_1')}. \quad (9)$$

If the value of N from (9) is greater than unity, then ϕ_2' corresponds to the point P_2; if it is less than unity, ϕ_2' corresponds to the point P_1 or P_3. In either case, it is not necessary to know which point ϕ_2' corresponds to in order to determine N. Having determined one equivalent circuit from (1) by means of (7)–(9), another one may be obtained by adding transmission lines of length $\pm(\pi/2)$ to θ_1 and θ_2 and choosing a new turns ratio $n^1 = 1/nZ_1$.

Other equivalent circuits also are readily found from a knowledge of the coefficients in the bilinear transformation given in (1). Two alternatives are illustrated in Fig. 3(a) and 3(b) and their parmaeters are given by

$$X_1 = -\frac{A}{C} \quad (10a)$$

$$X_2 = \frac{A}{C} - \frac{B}{D} \quad (10b)$$

$$n^2 Z_1 = \frac{DA}{C^2} - \frac{B}{C} \quad (10c)$$

for Fig. 3(a) and by

$$X_{11} = -\frac{B}{D} \quad (11a)$$

$$X_{22} = \frac{CZ_1}{D} \quad (11b)$$

$$X_{12} = \pm\sqrt{\frac{AZ_1}{D} - \frac{BCZ_1}{D^2}} \quad (11c)$$

for Fig. 3(b).

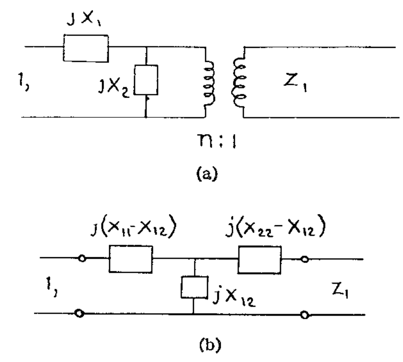

Fig. 3—Alternative equivalent circuits.

In the analysis of the equivalent circuit for slotted dielectric interfaces in free space,[2] as well as for dielectric steps in waveguides,[3] one obtains a bilinear transformation of the

* Received by the PGMTT, June 26, 1957.

[1] A. Weissflock, "Anwendung des transformersatzes über verlustlose vierpolen auf die hinter einander schaltung, von vierpolen," *Hochfreq. Tech. Elec. Akust.*, vol. 61, pp. 19–28; January, 1943.

[2] Paper in preparation by author.

[3] R. E. Collin and J. Brown, "Calculation of Equivalent Circuit of an Axially Unsymmetrical Waveguide Junction," IEE (London), Monograph No. 145R; August, 1955.

form given by (1). The above results are useful for determining the equivalent circuit parameters without the necessity of plotting the curve of ϕ_1 vs ϕ_2 and analyzing this curve by the standard procedures.

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Effect of a Mismatched Ring in a Traveling-Wave Resonant Circuit*

In traveling-wave resonant ring circuits wave amplitude "amplification" has been predicted and shown experimentally.[1-5] In the present note, the input reflection coefficient (input vswr) and wave amplification are considered when the resonant ring circuit contains a mismatch. Qualitatively a small mismatch (low vswr) can produce under resonant conditions a greatly "magnified" input vswr and reduces the maximum attainable amplification in the ring.

In Fig. 1 are shown resonant circuits consisting of ideal lossless directional couplers and a voltage mismatch Γ in each of the lossless waveguide rings. It should be noted that the designation of the output terminals of the directional couplers differ in the two circuits relative to the input terminals. Using scattering matrix notation,[6] the fundamental equations are:

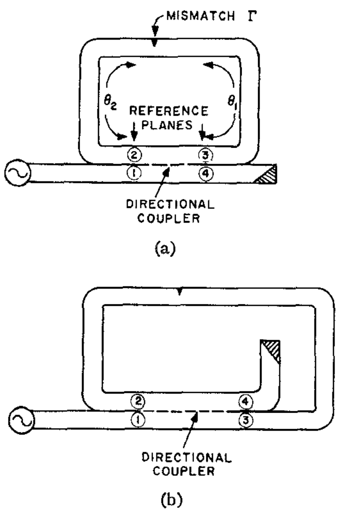

Fig. 1—Resonant ring circuits.

$$\begin{aligned} b_1 &= a_3S_c \\ b_2 &= a_3S_T \\ b_3 &= a_1S_c + a_2S_T \\ a_2 &= b_3\tau e^{-j(\theta_1+\theta_2)} + b_2\Gamma e^{-j2\theta_2} \\ a_3 &= b_3\Gamma e^{-j2\theta_1} + b_2\tau e^{-j(\theta_1+\theta_2)} \end{aligned} \quad (1)$$

where

$$S_T \equiv S_{13} = S_{31} = S_{24} = S_{42}$$

$$S_c \equiv S_{14} = S_{41} = S_{23} = S_{32}$$

$$|\tau| = \sqrt{1 - |\Gamma|^2}.$$

The input reflection coefficient, which exhibits a resonance phenomena when ϕ is varied, is given by:

$$\frac{b_1}{a_1} = \frac{S_c^2\Gamma e^{-j2\theta_1}}{1 + |S_T|^2e^{-j2\varphi} - 2|\tau S_T|e^{-j\phi}} \quad (2)$$

where

$$\phi \equiv \theta_1 + \theta_2 + \angle S_T + \angle\tau$$

$$\theta_\Gamma - \theta_\tau = \pm\frac{\pi}{2}.$$

If

$$|\Gamma| \leq \frac{|S_c|^2}{2 - |S_c|^2},$$

a single-peak resonance occurs when $\phi = 2\pi n$ radians, $n = 1, 2, 3, 4$, etc. and

$$\left|\frac{b_1}{a_1}\right| = \frac{|S_c^2\Gamma|}{1 + |S_T|^2 - 2|\tau S_T|}. \quad (3)$$

If

$$|\Gamma| \geq \frac{|S_c|^2}{2 - |S_c|^2} \quad \text{then} \quad \left|\frac{b_1}{a_1}\right| = 1$$

when:

$$\phi = \pm\cos^{-1}\left|\frac{\tau(1 + |S_T|^2)}{2S_T}\right|. \quad (4)$$

The two values of ϕ in (4) indicate double-peak resonance behavior.

Eq. (3) is plotted in Fig. 2 and shows that for small values of $|\Gamma|$, there is a reflection coefficient "magnification" of 37.9 and 5.8 for the 10- and 3-db couplers, respectively. For large values of $|\Gamma|$, the input vswr can become infinite at ring resonance, *i.e.*, when (4) is satisfied.

The wave amplification in the ring circuit is given by the following equation and is equivalent to that given by Tischer:[7]

$$\frac{b_3}{a_1} = S_c\frac{1 - |\tau S_T|e^{-j\phi}}{1 + |S_T|^2e^{-j2\varphi} - 2|\tau S_T|e^{-j\phi}}. \quad (5)$$

Eq. (5) exhibits a single-peak maximum when $\sin\phi = 0$ and a double-peak maxima when

$$\phi = \pm\cos^{-1}\frac{|\tau S_T| + \dfrac{1}{|\tau S_T|} - |\Gamma|\sqrt{\dfrac{1}{|\tau S_T|^2} + |S_T|^2 - 2}}{2}. \quad (6)$$

It should be pointed out that for very small values of $|\Gamma|$, only a single-peak resonance exists since $\cos\phi$ of (6) exceeds unity. The maximum values of $|\Gamma|$ which will yield single-peak resonance for 10-db and 3-db couplers are 0.026 and 0.177 respectively. To a very rough approximation these values of $|\Gamma|$ are about one-half of $|S_c|^2/(2-|S_c^2|)$. Operating under single-peak resonance conditions, the wave amplification $|b_3/a_1|$ as a function of $|\Gamma|$ is plotted in Fig. 3.

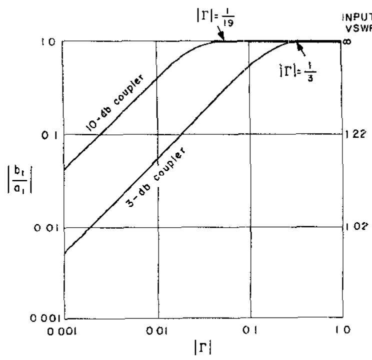

Fig. 2—Input reflection coefficient.

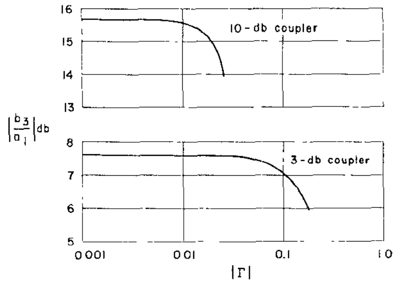

Fig. 3—Wave "amplification."

Another effect within the resonant ring due to the mismatch is on the ratio of incoming to outgoing waves at terminal 3 in Fig. 1. The analysis shows that with a 3-db coupler and when $\sin\phi = 0$,

$$\left|\frac{a_3}{b_3}\right| = \frac{\sqrt{2}|\Gamma|}{\sqrt{2} - |\tau|} \quad (7)$$

$$\doteq 3.42\,|\Gamma| \text{ for small values of } |\Gamma|.$$

Thus the vswr within the ring circuit is also "magnified." For a 10-db coupler the corresponding quantity is $19.5|\Gamma|$.

Referring to the annular-waveguide rotary joint,[2] the input vswr had a maximum value of about 3 (diagonal-arm condition) when the coupler "*A*" was about half open or approximately at 3-db coupling. These data would imply, referring to Fig. 2 above, that the mismatch vswr in the ring would be about 1.2 and this value appears entirely plausible.

In summary, a traveling-wave resonator requires very careful matching in the ring circuit if the input vswr is to be low and if maximum wave amplification is desired.

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* Received by the PGMTT, May 31, 1957.

[1] P. J. Sferrazza, "Traveling-wave resonator," *Tele-Tech.*, vol. 14, pp. 84–85, 142–143; November, 1955.

[2] K. Tomiyasu, "A new annular waveguide rotary joint," Proc. IRE, vol. 44, pp. 548–553; April, 1956.

[3] S. B. Cohn and F. S. Coale, "Directional channel-separation filters," Proc. IRE, vol. 44, pp. 1018–1024; August, 1956.

[4] F. S. Coale, "A traveling-wave directional filter," IRE Trans., vol. MTT-4, pp. 256–260; October, 1956.

[5] F. J. Tischer, "Resonance properties of ring circuits," IRE Trans., vol. MTT-5, pp. 51–56; January, 1957.

[6] E. W. Matthews, Jr., "The use of scattering matrices in microwave circuits," IRE Trans., vol. MTT-4, pp. 21–26; April, 1956.

[7] Tischer, *op. cit.*, (22).